JGP - Vol. 3, n. 4, 1986

Symplectic Twistor Spaces

IZU VAISMAN
University of Haifa - Dept. of Math.
Mt. Carmel
Haifa 31 999, Israel

Abstract. *The paper describes the geometry of the bundle* $\mathcal{T}(M, \omega)$ *of the compatible complex structures of the tangent spaces of an (almost) symplectic manifold* (M, ω)*, from the viewpoint of general twistor spaces* [3], [9], [1]. *It is shown that M has an either complex or almost Kaehler twistor space iff it has a flat symplectic connection. Applications of the twistor space* $\mathcal{T}$ *to the study of the differential forms of M, and to the study of mappings* $\varphi : N \to M$*, where N is a Kaehler manifold are indicated.*

INTRODUCTION

In the last few years general twistor spaces have been studied by several authors in connection with gauge fields theory and with the theory of harmonic mappings [3], [9], [1], [10], etc. The aim of the present paper is to apply this theory to symplectic geometry. We shall prove that the only symplectic manifolds which have an either integrable or almost Kaehler space of symplectic twistors are the locally flat symplectic manifolds. (See the exact formulation of the results in Theorem 2.1, 2.2, of this paper). We shall use the space of twistors in the description of differential forms, and in the study of mappings from a Kaehler manifold to a symplectic manifold.

1. DESCRIPTION OF SYMPLECTIC TWISTOR SPACES

Let (M^{2n}, ω) be an almost symplectic manifold (1) with the fundamental

Key-Words: Twistor spaces, symplectic geometry.
1980 Mathematics Subject Classification: *53 C 55, 58 F 05.*

(1) We are working in the C^∞-category.

2-form ω. A *symplectic twistor* on M is an almost complex tensor J at $x \in M$ such that

$$(1.1) \qquad \omega(JX, JY) = \omega(X, Y), \ \omega(X, JX) > 0 \text{ if } X \neq 0.$$

Then J has an associated almost hermitian metric

$$(1.2) \qquad g_J(X, Y) = \omega(X, JY).$$

The fibre bundle (e.g., [9])

$$(1.3) \qquad \pi : \mathscr{T} = \mathscr{T}(M, \omega) \to M$$

of all the symplectic twistors along M is called the *symplectic twistor space of M*.

We shall denote by $\mathscr{V}$ the *vertical foliation* of $\mathscr{T}$ by its fibers. The transverse bundle $T\mathscr{T}/\mathscr{V}$ is isomorphic to $E = \pi^{-1}(TM)$ and it plays a fundamental role for $\mathscr{T}$ [9]. Particularly E has the *tautological complex structures*

$$(1.4) \qquad \mathscr{C}_1 v = -Jv, \quad \mathscr{C}_2 v = Jv \quad (v \in E_J, \ J \in \mathscr{T}).$$

Furthermore, like in [9], we get

$$(1.5) \qquad \mathscr{V}_J = \{\mathscr{X} \in \operatorname{End} E_J / \mathscr{X}J + J\mathscr{X} = 0 \text{ and } \omega(\mathscr{X}v, w) + \omega(v, \mathscr{X}w) = 0\},$$

and we see that $\mathscr{V}$ has also a complex structure Φ given by

$$(1.6) \qquad \Phi_J \mathscr{X} = \mathscr{X}J \qquad (\mathscr{X} \in \mathscr{V}_J).$$

In order to combine (1.4) and (1.6) into almost complex structures of $\mathscr{T}$ we need a splitting of the exact sequence

$$(1.7) \qquad 0 \to \mathscr{V} \to T\mathscr{T} \to E \to 0,$$

and this is defined in twistor spaces theory [3], [9] by the *horizontal distribution* $\mathscr{H}$ defined on $\mathscr{T}$ by a connection ∇ on M. That is, for $J \in \mathscr{T}$, $\mathscr{H}_J$ is the plane of the tangent vectors to the paths obtained by ∇-parallel translations of J. In order that $\mathscr{H} \subset T\mathscr{T}$, we must take ∇ to be an almost symplectic connection, i.e.,

$$(1.8) \qquad \nabla\omega = 0.$$

Now every $\mathscr{X} \in T\mathscr{T}$ has a unique decomposition which we shall denote as

$$(1.9) \qquad \mathscr{X} = \mathscr{X}^h + \mathscr{X}^v \quad (\mathscr{X}^h \in \mathscr{H}, \mathscr{X}^v \in \mathscr{V}),$$

and $\mathscr{J}$ becomes endowed with the almost complex structures

$$(1.10) \qquad \mathscr{J}_1 = \mathscr{C}_1 + \Phi, \quad \mathscr{J}_2 = \mathscr{C}_2 + \Phi.$$

Moreover, E has a common hermitian metric associated with both $\mathscr{C}_1$ and $\mathscr{C}_2$

$$(1.11)\qquad \gamma_E(u,v)=\omega(\mathscr{C}_1 u, v)=\omega(u,\mathscr{C}_2 v),$$

which may be transfered to $\mathscr{H}$. This metric also induces a metric of End E, and therefore of $\mathscr{V}$ given by

$$(1.12)\qquad \gamma_{\mathrm{End}\,E}(\mathscr{X},\mathscr{Y})=\frac{1}{2}\,\mathrm{tr}\,[\gamma^{-1\,t}\mathscr{X}\gamma\mathscr{Y}]$$

where the righthand side contains the matrices of the respective elements with respect to arbitrary local bases of E. The direct sum of (1.11) and (1.12) yields an almost hermitian metric on $\mathscr{T}$ for the almost complex structures (1.10). We shall denote it by J, and we shall denote the corresponding Kaehler forms by

$$(1.13)\qquad \Xi^a(\mathscr{X},\mathscr{Y})=\mathscr{G}(\mathscr{I}_a\mathscr{X},\mathscr{Y})\qquad (a=1,2).$$

The restrictions of Ξ^a to horizontal vectors are exactly $\pi^*\omega$.

The structures introduced above have also a nice local description. Namely, let (e_i, e_{i*}) $(i=1,\dots,n;\ i^*=i+n)$ be a local field of symplectic bases of TM, i.e.,

$$(1.14)\qquad \omega(e_i,e_j)=\omega(e_{i*},e_{j*})=0,\qquad \omega(e_i,e_{j*})=\delta_{ij}.$$

Let $J\in\mathscr{T}$, and let

$$(1.15)\qquad J^+=\frac{1}{2}\,(\mathrm{Id.}-\sqrt{-1}\,J),\qquad J^-=\frac{1}{2}\,(\mathrm{Id.}+\sqrt{-1}\,J)$$

be the associated projectors onto the $\pm\sqrt{-1}$-eigenspaces of J. Then it is well known that J is defined by im J^+ which is a *positive* Lagrangian subspace of $(T^c_{\pi(J)}M,\omega)$ [2]. Positivity implies that im J^+ is transversal to span$\{e_i\}$, and then there is a unique symplectic transformation σ which preserves every e_i, and sends span $\{e_{i*}\}$ onto im J^+. σ can be expressed by [2]

$$(1.16)\qquad \sigma(e)=e,\qquad \sigma(e_*)=eZ+e_*;$$

where $e=(e_i)$, $e_*=(e_{i*})$ are one-line matrices, and $Z=X+\sqrt{-1}\,Y$ is a symmetric (n,n)-matrix with $Y>0$, i.e., en element of the *Siegel half-plane* [12].

Hence the fibre of $\mathscr{T}$ is the Siegel half-plane (it is the hermitian symmetric space $Sp(n)/U(n)$), and Z is a matrix complex coordinate along the fibers. By applying J to $\sigma(e_*)$ and $\overline{\sigma(e_*)}$, we can compute $J(e)$, $J(e_*)$, which yields the following representation of $J\in\mathscr{T}$

$$(1.17)\qquad J=\begin{pmatrix} XY^{-1} & -Y-XY^{-1}X \\ Y^{-1} & -Y^{-1}X \end{pmatrix}.$$

Using this formula, and taking $\mathrm{d}J$ for $\mathscr{X}$ of (1.6), we see that the complex structure Φ is the same as the complex structure defined by the complex coordinate Z.

Furthermore, if $((\epsilon^i), (\epsilon^{i*}))$ is the dual cobasis of (e, e_*) written as one-column vectors, it follows from (1.16) that

$$(1.18) \qquad \xi = \epsilon - Z\epsilon^*, \qquad \bar{\xi} = \epsilon - \bar{Z}\epsilon^*$$

define the bases of complex type (1, 0) for the dual bundle E^* with respect to $\mathscr{C}_1$ $\mathscr{C}_2$, respectively. Correspondingly

$$(1.19) \qquad \bar{f} = e\bar{Z} + e_*, \qquad f = eZ + e_*$$

are bases of im $\mathscr{C}_1^+$, im $\mathscr{C}_2^+$, respectively. We can also see (1.18) as a basis of either im $\mathscr{C}_2^+$, im $\mathscr{C}_1^-$ or im $\mathscr{C}_2^-$, im $\mathscr{C}_1^-$. With this interpretation, if $\mathscr{X} \in \mathscr{V}_J$, and $\mathscr{X} = J'(0)$ for a certain vertical path $J(t)$, the latter is given by

$$J(t)(\bar{f}(t)) = \sqrt{-1}\,\bar{f}(t), \; J(t)(f(t)) = -\sqrt{-1}\,f(t),$$

and we get by a derivation

$$(1.20) \qquad \mathscr{X}(\bar{f}) = J'(0)(\bar{f}) = \sqrt{-1}\,f'(0) - J(f'(0)), \; \mathscr{X}(f) = \overline{\mathscr{X}(\bar{f})}.$$

Here $\bar{f}'(0)$ is computable by (1.19), and we see that $\mathscr{X} \in \mathscr{V}_J$ can be represented by

$$(1.21) \qquad \mathscr{X}(\bar{f}) = -fY^{-1}\mathrm{d}\bar{Z}, \quad \mathscr{X}(f) = -\bar{f}Y^{-1}\mathrm{d}Z.$$

Now, in order to obtain the metrics on E and $\mathscr{V}$, we shall notice that (1.18) implies

$$(1.22) \qquad \pi^*\omega = -\frac{\sqrt{-1}}{2}\,{}^t\xi \wedge Y^{-1}\xi = \frac{\sqrt{-1}}{2}\,{}^t\xi \wedge Y^{-1}\xi.$$

Hence the metric (1.11) is the hermitian form

$$(1.23) \qquad \gamma_E = {}^t\xi \otimes Y^{-1}\bar{\xi} = {}^t\bar{\xi} \otimes Y^{-1}\xi.$$

Furthermore, by computing (1.12) with respect to the basis (1.19) and by using (1.21), we get that $\mathscr{G}$ induces on $\mathscr{V}$ the metric

$$(1.24) \qquad g_{\mathscr{V}} = \mathrm{tr}\{Y^{-1}\mathrm{d}Z \otimes Y^{-1}\mathrm{d}\bar{Z}\},$$

which is precisely the metric studied by Siegel [12]. It can also be checked via (1.12) that

$$(1.25) \qquad g_{\mathscr{V}_J}(\mathscr{X}, \mathscr{Y}) = \sum_{i=1}^{n} \{\omega(\mathscr{X}e_{i*}, \mathscr{Y}e_i) - \omega(\mathscr{X}e_i, \mathscr{Y}e_{i*})\},$$

where $\mathscr{X}, \mathscr{Y} \in \mathscr{V}_J$.

Finally, let us consider the connection ∇, and write down its local equations in the matrix form

$$(1.26)\qquad (\nabla e\ \nabla e_*) = (e\ e_*)\begin{pmatrix}\theta & \kappa \\ \lambda & \mu\end{pmatrix}$$

where the (2, 2)-matrix of connection forms takes values in the symplectic Lie algebra $sp(n)$, i.e.,

$$(1.27)\qquad {}^t\kappa = \kappa, \quad {}^t\lambda = \lambda, \quad {}^t\theta + \mu = 0.$$

Then the parallel translation of J is equivalent to the parallel translation of the distribution span$(\sigma(e_*))$ of (1.16), and of its conjugated distribution, and, by (1.26) the equations of these translations, i.e., the equations of $\mathscr{H}$ will be

$$(1.28)\qquad \zeta = 0, \qquad \bar{\zeta} = 0,$$

where

$$(1.29)\qquad \zeta = dZ + \theta Z + Z{}^t\theta + \kappa - Z\lambda Z.$$

It follows that $(\epsilon, \epsilon^*; \zeta, \bar{\zeta})$ is a local cobasis of $\mathscr{T}$ adapted to the splitting (1.9), and since pullback ζ to $\mathscr{V} = dZ$, we see that (ξ, ζ) is a cobasis of the forms of the complex type (1, 0) of $\mathscr{I}_1$, and $(\bar{\xi}, \zeta)$ is a similar basis of $\mathscr{I}_2$, defined by (1.10). It also follows from (1.23), (1.24) that the corresponding almost hermitian metric $\mathscr{G}$ on $\mathscr{T}$ is

$$(1.30)\qquad \mathscr{G} = {}^t\xi \otimes Y^{-1}\bar{\xi} + \operatorname{tr}\{Y^{-1}\zeta \otimes Y^{-1}\bar{\zeta}\},$$

and the associated Kaehler forms (1.13) are, respectively,

$$(1.31)\qquad \Xi^{1,2} = \pi^*\omega \mp \frac{\sqrt{-1}}{2}\{\operatorname{tr}\ Y^{-1}\zeta \wedge Y^{-1}\bar{\zeta}\}.$$

Another important piece of structure of a twistor space is the induced connection $\nabla' = \pi^{-1}\nabla$ of E. Using it, and following [9], we define on E the connection

$$(1.32)\qquad D_{\mathscr{X}} u = \nabla'_{\mathscr{X}} u + \frac{1}{2}\mathscr{X}^v J u \qquad (\mathscr{X} \in T\mathscr{T}, u \in E).$$

Using (1.19), (1.21) and (1.26), we obtain for D the local equations

$$(1.33)\qquad D\bar{f} = \bar{f}\left(\lambda\bar{Z} + \mu + \frac{\sqrt{-1}}{2}\ Y^{-1}\bar{\zeta}\right), Df = f\left(\lambda Z + \mu - \frac{\sqrt{-1}}{2}\ Y^{-1}\zeta\right),$$

which proves that D preserves the complex structures $\mathscr{C}_1$, $\mathscr{C}_2$ of E. A consequence of this fact is

$$\nabla'_{\mathscr{X}} \mathscr{C}_1 = -\mathscr{X}^v, \qquad \nabla'_{\mathscr{X}} \mathscr{C}_2 = \mathscr{X}^v. \tag{1.34}$$

Now, on one hand, D may be transposed to the horizontal bundle $\mathscr{H}$. On the other hand, D induces a connection on End E which commutes with J, hence D induces a connection on $\mathscr{V}$ which commutes with the complex structure Φ of $\mathscr{V}$. By adding up these facts, we see that D extends to an almost complex connection on $\mathscr{T}$ with respect to both structures $\mathscr{J}_1$, $\mathscr{J}_2$ of (1.10). Moreover, it is easy to see that $\nabla\omega = 0$ implies that $D(\pi^*\omega) = 0$, and D also preserves the almost hermitian metric $\mathscr{G}$ of $\mathscr{T}$.

Generally, D has a torsion T^D. Its computation is a technical matter (see [9]), and it gives

$$[T^D(\mathscr{X}, \mathscr{Y})]^h = \text{horizontal lift} \left\{ T^\nabla(X, Y) + \frac{1}{2}\, \mathscr{X}^v JY - \frac{1}{2}\, \mathscr{Y}^v JX \right\}, \tag{1.35}$$

$$[T^D(\mathscr{X}, \mathscr{Y})]^v Z = \text{horizontal lift} \{R^\nabla(X, Y)(JZ) - JR^\nabla(X, Y)Z\}, \tag{1.36}$$

where T^∇, R^∇ are the torsion and curvature of ∇ respectively, $\mathscr{X}$, $\mathscr{Y} \in T_J\mathscr{T}$, $Z \in E_J$, $X = \pi_*\mathscr{X}$, $Y = \pi_*\mathscr{Y}$. (In this computation one uses extensions of $\mathscr{X}$, $\mathscr{Y}$, Z to fields with horizontal parts projectable to M, and one uses the commutation of D with horizontal and vertical projections, and the interpretation $\mathscr{X}^v = \nabla'_{\mathscr{X}} J$ of the vertical component of $\mathscr{X}$ which is given by (1.34).

2. INTEGRABILITY AND THE KAEHLER CONDITION

It is obvious that the integrability of the almost complex structures (1.10) is an important problem. Here, we apply the general results of [9] in order to discuss the symplectic case. The Nijenhuis tensor of these structures [7] can be written as [9]

$$N^{\mathscr{J}_a}(\mathscr{X}, \mathscr{Y}) = -8\, Re\, \mathscr{J}_a^+ T^D(\mathscr{J}_a^- \mathscr{X}, \mathscr{J}_a^- \mathscr{Y}) \quad (a = 1, 2), \tag{2.1}$$

and the structure $\mathscr{J}_a$ is integrable iff $N^{\mathscr{J}_a} = 0$. Since the structure is real this means

$$\mathscr{J}_a^+ T^D(\mathscr{J}_a^- \mathscr{X}, \mathscr{J}_a^- \mathscr{Y}) = 0. \tag{2.2}$$

For $a = 2$, if we evaluate the horizontal part of (2.2) for $\mathscr{X} \in \mathscr{H}$, $\mathscr{Y} \in \mathscr{V}$ by using (1.35), we get necessarily $\mathscr{Y} = 0$. But for $\dim M > 0$ nonzero vertical vectors exist. Hence $\mathscr{J}_2$ is never integrable. For $a = 1$, (2.2) has a horizontal and a vertical component, and we get the integrability conditions [3], [9] (2.3) $$J^{+} T^{\nabla}(J^{-}X, J^{-}Y) = 0,$$

(2.4) $$J^{+} R^{\nabla}(J^{-}X, J^{-}Y)(J^{-}Z) = 0,$$

for every $J \in \mathscr{T}$ and $X, Y, Z \in T_{\pi(J)}M$. These conditions can be analyzed by group representation considerations, which give T^{∇} and R^{∇} [9].

In the symplectic case, we get

2.1. THEOREM. *Let (M^{2n}, ω) be an almost symplectic manifold. Then M has a connection ∇ which induces an integrable structure $\mathscr{J}_1$ on $\mathscr{T}(M, \omega)$ iff either $n = 1$ or $n > 1$ and M is locally conformally symplectic flat. Particularly, if M is symplectic the last condition means that M is symplectic flat.*

Proof. In [9], it is shown that (2.3) holds iff

(2.5) $$T^{\nabla}(X, Y) = \alpha(X)Y - \alpha(Y)X$$

for some 1-form α. Accordingly, using $\nabla\omega = 0$ we obtain

(2.6) $$d\omega(X, Y, Z) = \sum_{\text{Cycl.}} \omega(T^{\nabla}(X, Y), Z) = 2(\alpha \wedge \omega)(X, Y, Z).$$

Conversely, if (2.6) holds, the connection

(2.7) $$\nabla_X Y = \mathring{\nabla}_X Y + P(X, Y),$$

where $\mathring{\nabla}$ is an arbitrary linear connection, and P is the tensor defined by

(2.8) $$\omega(P(X, Y), Z) = \frac{1}{2}(\mathring{\nabla}_X\omega)(Y, Z) + \frac{1}{6}(\mathring{\nabla}_Y\omega)(X, Z) + \frac{1}{6}(\mathring{\nabla}_Z\omega)(X, Y),$$

satisfies (1.8) and it has the torsion (2.5). (See, for instance, [18]). Hence (2.5) can be achieved iff (2.6) holds good. For $n = 1$, this holds with $\alpha = 0$. For $n = 2$, a 1-form α satisfying (2.6) always exists [8]. For $n > 2$, if (2.6) holds for some α then, necessarily, $d\alpha = 0$ and $2\alpha = d\sigma$ for some local functions σ. Then $d(e^{-\sigma}\omega) = 0$, and we see that M is a locally conformal symplectic manifold (e.g., [17]).

Furthermore, if (2.5) holds, the first Bianchi identity for linear connections with torsion becomes

(2.9) $$\sum_{\text{Cycl.}} R^{\nabla}(X, Y)Z = \sum_{\text{Cycl.}} [d\alpha(X, Y)]Z.$$

If $d\alpha = 0$, we may go on with the result of [9]. In order to include the case

$n = 2$, $d\alpha \neq 0$, we take

$$\rho(X, Y)Z = R^{\nabla}(X, Y)Z - d\alpha(X, Y)Z \tag{2.10}$$

which satisfies the torsionless Bianchi identity, and R^{∇} satisfies (2.4) iff ρ satisfies (2.4). Then ρ must have the form given for R^{∇} in [9], and we get

$$R^{\nabla}(X, Y)Z = \mu(X, Y)Z - \mu(Y, X)Z + \mu(X, Z)Y - \mu(Y, Z)X + d\alpha(X, Y)Z, \tag{2.11}$$

for some tensor μ.

Now, take the covariant curvature tensor

$$S^{\nabla}(U, Z, X, Y) = \omega(R^{\nabla}(X, Y)Z, U) \tag{2.12}$$

which is symmetric in U, Z (e.g., [18]). If we use local natural components, and (2.11), the symmetry of S means

$$2\nu_{\alpha\beta}\omega_{\gamma\lambda} = \mu_{\alpha\lambda}\omega_{\beta\gamma} - \mu_{\beta\lambda}\omega_{\alpha\gamma} + \mu_{\beta\gamma}\omega_{\alpha\lambda} - \mu_{\alpha\gamma}\omega_{\beta\lambda}, \tag{2.13}$$

where

$$\nu(X, Y) = \mu(X, Y) - \mu(Y, X) + d\alpha(X, Y). \tag{2.14}$$

Furthermore, define $\omega^{\lambda\gamma}$ by $\omega_{\alpha\lambda}\omega^{\lambda\gamma} = \delta^{\gamma}_{\alpha}$, and contract (2.13) by $\omega^{\lambda\gamma}$. This gives

$$2n\nu_{\alpha\beta} = \mu_{\beta\alpha} - \mu_{\alpha\beta}. \tag{2.15}$$

If this result is inserted in (2.13), and then we contract by $\omega^{\beta\gamma}$, we get

$$n\kappa\omega_{\alpha\beta} = \mu_{\lambda\alpha} + \mu_{\alpha\lambda}(2n^2 - 2n - 1) \quad (\kappa = \omega^{\beta\gamma}\mu_{\beta\gamma}). \tag{2.16}$$

Since ω is skew-symmetric, (2.16) yields

$$2n(n-1)(\mu_{\alpha\lambda} + \mu_{\lambda\alpha}) = 0. \tag{2.17}$$

Let us assume $n > 1$. Then μ is skew-symmetric, (2.16) contracted by $\omega^{\alpha\lambda}$ yields $\kappa = 0$, and using again (2.16) we get $\mu = 0$. Now, (2.15) and (2.14) imply $\nu = 0$ and $d\alpha = 0$.

Hence, for $n > 1$, (2.11) holds iff (M, ω) is a locally conformal symplectic manifold, and $R^{\nabla} = 0$. (If $d\alpha = 0$, then $R^{\nabla} = 0$ implies $\mu = 0$ by the same computation as above). Let us assume that these conditions hold good, and that we have (2.6) with $2\alpha = d\sigma$ locally. Then a straightforward computation shows that the connection

$$\tilde{\nabla}_X Y = \nabla_X Y - \alpha(X)Y \tag{2.18}$$

is torsionless, it has the same curvature as ∇, i.e., $\tilde{R} = R = 0$, and $\tilde{\nabla}_X(e^{-\sigma}\omega) = 0$. This is precisely the meaning of the fact that (M, ω) is locally conformally symplectic flat.

Conversely, if (M, ω) is locally conformally symplectic flat as above, we have an open covering $M = \cup U_s$ with the local functions $\sigma_s : U_s \to R$ such that $2\alpha = d\sigma_s$, and with local torsionless flat linear connections $\tilde{\nabla}^s$ such that $\tilde{\nabla}^s(e^{-\sigma_s}\omega) = 0$. Then, over $U_s \cap U_{s'}$ we have $\sigma_{s'} - \sigma_s = \text{const.}$, hence, also, $\tilde{\nabla}^{s'}(e^{-\sigma_s}\omega) = 0$. This implies some relation

$$\tilde{\nabla}^{s'}_X Y = \nabla^s_X Y + A^{ss'}(X, Y), \tag{2.19}$$

where $\omega(A^{ss'}(X, Y), Z)$ is completely symmetric (e.g., [18]). On the other hand, we get

$$\tilde{\nabla}^s_X \omega = 2\alpha(X)\omega = \tilde{\nabla}^{s'}_X \omega, \tag{2.20}$$

and, using (2.19), this implies $A^{ss'} = 0$. Therefore, the local connections $\tilde{\nabla}^s$ glue up to a global connection $\tilde{\nabla}$ on M. Then, the connection ∇ associated to $\tilde{\nabla}$ by (2.18) satisfies $\nabla\omega = 0$, and it has torsion (2.5) and vanishing curvature, and provides an integrable structure $\mathscr{J}_1$ on $\mathscr{T}(M, \omega)$.

Now, in the case $n = 1$, $d\omega = 0$, and M has torsionless symplectic connections. Following [9], if (2.5) is satisfied for some ∇, this ∇ may be changed to a torsionless symplectic connection without changing the complex structure $\mathscr{J}_1$. If this is done, the tensor (2.12) has the expression [18]

$$S^\nabla(U, Z, X, Y) = \rho^\nabla(U, Z)\omega(X, Y), \tag{2.21}$$

where ρ^∇ is the Ricci curvature of ∇. Let us take in (2.11) $\mu = -\rho^\nabla$. Then, in covariant form, (2.11) is equivalent to

$$\rho^\nabla(U, Z)\omega(X, Y) = \rho^\nabla(Y, Z)\omega(X, U) - \rho^\nabla(X, Z)\omega(Y, U), \tag{2.22}$$

and this condition is always satisfied as it can be easily checked on a symplectic basis e_1, e_{1*}.

This completes the proof of Theorem 2.1. The last assertion of the theorem is clear from this proof. ∎

Another interesting result for symplectic twistor spaces is

2.2. THEOREM. *Let (M, ω) be an almost symplectic manifold. Then $(\mathscr{T}(M), \mathscr{J}_a, \mathscr{G})$ $(a = 1, 2)$ are simultaneously almost Kaehler manifolds, and this situation occurs iff M is symplectic, and it has a connection ∇ with vanishing curvature and such that $\nabla\omega = 0$.*

Proof. The Kaehler condition considered is $d\Xi^a = 0$ for Ξ^a defined by (1.31), and if we express it on the various possible horizontal and vertical arguments we see that $d\Xi^a = 0$ iff

(2.23) $$d\omega = 0, \qquad d\Omega = 0,$$

where Ω is the second term of (1.31), and it defines the Kaehler form of $\mathscr{V}$. This proves our first assertion, and the fact that M must be a symplectic manifold.

Furthermore, we have

(2.24) $$d\Omega(\mathscr{X},\mathscr{Y},\mathscr{Z}) = \sum_{\text{Cycl.}} \Omega(T^D(\mathscr{X},\mathscr{Y}),\mathscr{Z}) \; (\mathscr{X},\mathscr{Y},\mathscr{Z} \in T\mathscr{T}).$$

and, in view of the definition of Ω and of (1.36), $d\Omega = 0$ iff

(2.25) $$\Omega([T^D(\mathscr{X},\mathscr{Y})]^v, \mathscr{Z}^v) = 0$$

for every $\mathscr{X}$, $\mathscr{Y} \in \mathscr{H}$ and $\mathscr{Z}^v \in \mathscr{V}$. Since Ω is nondegenerate along $\mathscr{V}$, (2.25) is equivalent to $[T^D(\mathscr{X},\mathscr{Y})]^v = 0$, i.e., by (1.36), to

(2.26) $$R^\nabla(X, Y) \circ J = J \circ R^\nabla(X, Y)$$

for every $J \in \mathscr{T}$; $X, Y \in TM$. Finally, by means of the tensor S of (2.12), our condition becomes

(2.27) $$S(U, JZ, X, Y) + S(JU, Z, X, Y) = 0.$$

Now, since for every symplectic tangent basis (e_i, e_{i*}) there exists $J \in \mathscr{T}$ such that $Je_i = e_{i*}$, and since S is symmetric with respect to the first two arguments, we see that (2.27) implies

(2.28) $$S(e_i, e_{j*}, X, Y) = 0, S(e_i, e_j, X, Y) = S(e_{i*}, e_{j*}, X, Y).$$

The same relations hold with respect to a new symplectic basis $(e_i, te_i + e_{i*})$ with $t \neq 0$. This implies $S = 0$ necessarily, which ends the proof of Theorem 2.2. ■

Notice that we do not ask ∇ to have zero torsion.

Let us also notice the following consequence of Theorems 1.1 and 1.2

2.3. COROLLARY. *If the symplectic twistor bundle $\mathscr{T}(M, \omega)$ of a symplectic manifold (M, ω) has no complex structure or no symplectic structure then (M, ω) has no flat symplectic connection.*

Another result which we should like to mention here is

2.4. PROPOSITION. *Let (M, ω) be an almost symplectic manifold. Then different symplectic connections yield different almost complex structures on $\mathscr{T}(M, \omega)$.*

Proof. Following [3] the condition for two connections related by

(2.29) $$\tilde{\nabla}_X Y = \nabla_X Y + K(X, Y)$$

to define the same structure $\mathscr{I}_1$ is

(2.30) $$J^+ K(J^- X, J^- Y) = 0.$$

For (2.30), if we apply the group representation analysis of [9] we obtain (see also [3])

(2.31) $$K(X, Y) = \alpha(X) Y + \beta(Y) X$$

for some 1-forms α, β. Furthermore, since $\nabla\omega = \tilde{\nabla}\omega = 0$ we must have [18]

(2.32) $$\omega(K(X, Y), Z) = \omega(K(X, Z), Y).$$

For K of (2.31), and after some contractions, (2.32) implies $\alpha = \beta = 0$, i.e., $K = 0$.

In a similar way the condition to have the same structure $\mathscr{I}_2$ is

(2.33) $$J^+ K(J^+ X, J^- Y) = 0$$

which is equivalent to

(2.34) $$B(JX, JY, Z) + B(JX, Y, JZ) - B(X, JY, JZ) + B(X, Y, Z) = 0,$$

where $B(X, Y, Z) = \omega(K(X, Y), Z)$, and it satisfies (2.32). Now, we shall consider the family of symplectic bases $(e_i, e_{i*}(t) = te_i + e_{i*})$ $(t \neq 0)$ and $Je_i = e_{i*}(t)$, and we shall explicitate (2.34) for this J, and for $X = e_i$, $Y = e_j$, $Z = e_k$, then for $X = Je_i$, $Y = Je_j$, $Z = Je_k$. As a result we see that we must have

(2.35) $$\begin{aligned} &B(e_i, e_j, e_k) = B(e_{i*}, e_j, e_k) = B(e_i, e_{j*}, e_{k*}) = \\ &\qquad = B(e_{i*}, e_{j*}, e_{k*}) = 0, \\ &B(e_{i*}, e_{j*}, e_k) + B(e_{i*}, e_j, e_{k*}) = 0, \\ &B(e_i, e_j, e_{k*}) + B(e_i, e_{j*}, e_k) = 0. \end{aligned}$$

Furthermore, if we take the more ample family of symplectic bases $(e_i, \theta\lambda^u_{i*} e_u + e_{i*})(\lambda^u_{i*} = \lambda^i_{u*}, \theta \neq 0)$, we get similarly from (2.34), modulo (2.35), that

(2.36) $$\begin{aligned} &\lambda^v_{j*} B(e_{i*}, e_v, e_{k*}) + \lambda^s_{k*} B(e_{i*}, e_{j*}, e_s) = 0, \\ &\lambda^u_{i*}\lambda^v_{j*} B(e_u, e_v, e_{k*}) + \lambda^u_{i*}\lambda^s_{k*} B(e_u, e_{j*}, e_s) = 0. \end{aligned}$$

Here, in the first relation we take $j = k$, and in the second $i = j = k$. Then, since one line of the symmetric matrix λ can be taken arbitrarily, we shall obtain

(2.37) $$B(e_u, e_v, e_{k*}) = B(e_{i*}, e_{k*}, e_v) = 0,$$

which ends the proof of Proposition 2.4.

We shall end this section by a closer look at the case $n = 1$. Then M is a surface, ω is a volume element, and

$$\mathcal{T}_x = \{J \in \operatorname{End} T_x M \mid J^2 = -I,\ \omega\text{-orientation} = J\text{-orientation}\} \tag{2.38}$$

($x \in M$), i.e., $\mathcal{T}(M, \omega) = \mathcal{T}(M^{or})$ depends only of the orientation of M. The fibre $\mathcal{T}_x$ is the upper half plane of a complex variable z, i.e., a hemisphere of the unit sphere S^2.

From Theorem 1.1, it follows that we can construct a complex structure on $\mathcal{T}$ in the following way. We take a riemannian metric g with volume from ω, and use semigeodesic local coordinates (u, v) such that g is

$$ds^2 = du^2 + G(u, v)dv^2 \qquad (G > 0), \tag{2.39}$$

and therefore

$$\omega = \sqrt{G}\, du \wedge dv. \tag{2.40}$$

Then $\epsilon = du$, $\epsilon^* = \sqrt{G}\, dv$ is a symplectic cobasis, and $e = \partial/\partial u$, $e_* = (1/\sqrt{G}) \cdot$ $\cdot (\partial/\partial v)$ is the dual symplectic basis. Now we may use the Levi-Civita connection of M which is torsionless and symplectic, and it has the local equations

$$\nabla e_1 = [(\sqrt{G})_u dv]\, e_2, \qquad \nabla e_2 = -[(\sqrt{G})_u dv]\, e_1, \tag{2.41}$$

where the index u denotes $\partial/\partial u$. According to the general formulas (1.18), (1.29), $\mathcal{T}(M)$ has a complex structure with the basis of (1, 0)-forms

$$\xi = du - z\sqrt{G}\, dv, \qquad \zeta = dz - (1 + z^2)(\sqrt{G})_u\, dv. \tag{2.42}$$

Correspondingly, we have on $\mathcal{T}(M)$ the hermitian metric $\mathcal{G}$ of (1.30) with the Kaehler form (1.31), i.e.,

$$\Xi^1 = \sqrt{G}\, du \wedge dv - \frac{1}{y^2}\{dx \wedge dy + (\sqrt{G})_u\, dv \wedge \wedge [2xy\, dx + (1 + x^2 - y^2) dy]\}, \tag{2.43}$$

where $z = x + \sqrt{-1}\, y$ $(y > 0)$. This gives

$$d\Xi^1 = -K[2xy\, dx + (1 + x^2 - y^2) dy] \wedge \Xi^1, \tag{2.44}$$

and we see that we have a Kaehler structure on $\mathcal{T}(M)$ iff $K = 0$, and we have a locally conformal Kaehler structure [14] iff $K = \text{const}$. ∎

3. POTENTIAL APPLICATIONS

The geometric applications of symplectic twistor spaces are still to be discovered and studied. Here, we make only a few introductory remarks about this subject.

a. A first idea is to use the manifold $\mathscr{T}$ of Section 1 in order to derive properties of differential forms of the almost symplectic manifold (M, ω). It is clear that the algebra of these forms $\wedge M$ can be seen as a subalgebra of the algebra of cross-sections $\wedge E^*$ which is built over a hermitian vector bundle, and it has therefore the well-known corresponding algebraic operators and properties [19]. If we agree to call *projectable* to those elements of $\wedge E^*$ which belong to $\wedge M$, and if we discuss the projectability of the algebraic operators mentioned above, we shall refind the operators and properties of $\wedge M$ as given in [8].

Namely, we have the operator [19]

$$C\alpha(X_1, \ldots, X_k) = \alpha(\mathscr{C}_1 X_1, \ldots, \mathscr{C}_1 X_k) \qquad (\alpha \in \wedge^k E^*). \tag{3.1}$$

For $k = 1$, $\# C$ (where $\# : E^* \approx E$ is defined by the metric γ_E of (1.11) is the isomorphism $E^* \approx E$ defined by ω [8]. If $*$ is the Hodge star of γ_E where the volume element of E is taken to be $(-1)^n \omega^n/n!$, then a simple computation shows that $*C$ preserves projectability, and it is precisely the operator $\tilde{*}$ of [8]. The operators $L\alpha = \omega \wedge \alpha$, and

$$\Lambda = *^{-1} L * = \tilde{*}^{-1} L \tilde{*} \tag{3.2}$$

of [19] and [8] preserve projectability. Because of the hermitian structure we have (e.g., [19]) a unique decomposition

$$\alpha = \sum_{h \geq (k-n)^+} L^h \alpha_h, \qquad \alpha \in \wedge^k E^*, \tag{3.3}$$

where $(k-n)^+ = \max(0, k-n)$, and $\alpha_h = \Phi_{k,h}(L, \Lambda)\, \alpha \in \ker \Lambda$, where $\Phi_{k,h}(L, \Lambda)$ are polynomials in L, Λ which do not depend on α. Hence, if α is projectable so are α_h, and (3.3) becomes a theorem [8] for $\wedge M$, which needs no independent proof.

Furthermore, we may see $\wedge E^*$ as isomorphic to $\wedge \mathscr{H}^*$, i.e., the algebra of forms on $\mathscr{T}$ which are of type $(\cdot\,, 0)$ with respect to the $(\mathscr{H}, \mathscr{V})$-decomposition (1.9) of $T\mathscr{T}$; we call this the $\mathscr{V}$-type of a form. It is well-known [13] that one has a decomposition of the exterior differential on $\mathscr{T}$ as

$$d = d'_{(1,0)} + d''_{(0,1)} + \partial_{(2,-1)} \tag{3.4}$$

where the indices denote the $\mathscr{V}$-type of the operators. Then we have

$$\Lambda M = (\Lambda \mathscr{H}^*) \cap (\ker d''), \tag{3.5}$$

and $d_M = d'$. The codifferential $\delta' = - * d' *$ on $\mathscr{T}$ doesn't preserve projectability, but $\tilde{\delta}' = C\delta' C$ [8] does. As in [19], we have

$$L d' = d' L, \qquad \Lambda \delta' = \delta' \Lambda, \tag{3.6}$$

in the symplectic case (i.e., $d\omega = 0$). Since C commutes with Λ, we also get [8]

$$\Lambda \tilde{\delta}' = \tilde{\delta}' \Lambda. \tag{3.7}$$

Then, by the proof in [19], we get [8]

$$\Lambda d' - d' \Lambda = - C^{-1} \delta' C = (-1)^{\deg} \tilde{\delta}', \tag{3.8}$$

$$L \delta' - \delta' L = C^{-1} d' C = (-1)^{p+1} C d' C. \tag{3.8'}$$

Finally, we may consider the (non-elliptic) Laplacian $\Delta' = d'\delta' + \delta' d'$ which will commute with $*$, d, L, such that for $\alpha \in \ker \Delta'$, α_h of (3.3) also belong to $\ker \Delta'$. Δ' doesn't preserve projectability, but $\tilde{\Delta}' = - C\Delta' C = d' \tilde{\delta}' + \tilde{\delta}' d'$ does [8]. A usual computation shows that d', δ' are formal adjoint operators, and Δ' is self-adjoint for the scalar product

$$\langle \alpha, \beta \rangle = \int_{\mathscr{T}} \alpha \wedge * \beta \wedge \Xi \tag{3.9}$$

(where α, β are compactly supported forms of $\mathscr{V}$-type $(\cdot, 0)$ on $\mathscr{T}$, and Ξ is the volume form along $\mathscr{V}$), if the condition $d' \Xi = 0$ holds good. Etc.

Another possibility to relate forms on $\mathscr{T}$ and on M is by fibre integration along the fibers of $\pi : \mathscr{T} \to M$. It is known [5] that, if M is compact, then

$$\int : \Lambda_c^{pN} \mathscr{T} \to \Lambda^p M \tag{3.10}$$

(where Λ_c^{pN} denotes compactly supported forms of $\mathscr{V}$-type (p, N), and $N = n(n+1)$) is a surjection. Generally, on Λ_c^{pN} we have $d = d' + \partial$, hence we do not get a cochain subcomplex of the de Rham complex of $\mathscr{T}$, but, using the $\mathscr{V}$-type homogeneous consequences of $d^2 = 0$ [13], it is easy to see that such a subcomplex $\mathscr{K}_N$ is defined by the subspaces

$$\mathscr{K}_N^p = \{\alpha \in \Lambda_c^{pN} \mathscr{T} / \partial\alpha = 0\}. \tag{3.11}$$

Accordingly, and since integration along fibers commutes with d, we deduce the existence of a homomorphism

$$\int : H^p(\mathscr{K}_N) \to H^p(M, R). \tag{3.12}$$

If M is a symplectic flat manifold, $\mathscr{H}$ is integrable, $\partial = 0$ and $\mathscr{K}_N = \Lambda_c^{pN}\mathscr{T}$. It is also known [5] that

$$\mathscr{P}_N(\alpha, f\beta) = \mathscr{P}_{\mathscr{T}}(\pi^*\alpha, \beta), \tag{3.13}$$

where $\alpha \in \Lambda^p M$, $\beta \in \Lambda_c^{n-p,N}\mathscr{T}$, π is the projection $\mathscr{T} \to M$, and $\mathscr{P}$ is the global scalar product which leads to Poincaré duality. It follows from (3.13) that, at the cohomology level, if π^* is injective, the $\int$ is surjective. In our case this is true since the bundle $\mathscr{T}$ has global cross-sections. This yields

3.1. PROPOSITION. *Let M be a compact symplectic flat manifold. Then, there exists a cohomology epimorphism*

$$\int : H^p(\Lambda_c^{*N}\mathscr{T}) \to H^p(M, R). \tag{3.14}$$

Finally, it should also be remarked that, in case $\mathscr{T}$ has a complex structure, we might also use the decomposition of forms of M into terms of homogeneous complex type of $\mathscr{T}$. The terms will not be projectable, but the decomposition may be interesting at least since M itself may have no complex structure. (See [4] for examples of symplectic flat manifolds with no complex structure).

b. A second potential application to be considered is to the computation of the Chern classes of an almost symplectic manifold (see, e.g., [16]). Indeed, we may use connection D of (1.32) in order to compute the Chern classes of $(E, \mathscr{C}_2)$, and then pull back these classes from $\mathscr{T}$ to M by means of a global crossection $J : M \to \to \mathscr{T}$.

Namely, let ∇ be defined locally by (1.26), and let

$$\begin{pmatrix} \Theta & K \\ \Lambda & -{}^t\Theta \end{pmatrix} = \begin{pmatrix} d\theta + \theta \wedge \theta + \kappa \wedge \lambda & d\kappa + \theta \wedge \kappa + \kappa \wedge \mu \\ d\lambda + \mu \wedge \lambda + \lambda \wedge \theta & d\mu + \mu \wedge \mu + \lambda \wedge \kappa \end{pmatrix} \tag{3.15}$$

be the corresponding curvature forms. Then, connection D has the local complex equations given by the second part of formula (1.33), and its curvature will be

$$\begin{aligned} \Phi^D = \Lambda Z - {}^t\Theta - \frac{\sqrt{-1}}{2}\, Y^{-1}(\Theta Z + K + Z^t\Theta - Z\Lambda Z) - \\ -\left(\lambda + \frac{1}{4}\, Y^{-1}\zeta Y^{-1}\right)\wedge \zeta - \\ -\frac{\sqrt{-1}}{2}\,(dY^{-1} - Y^{-1}\theta + Y^{-1}Z\lambda + \lambda Z Y^{-1} + \mu Y^{-1}) \wedge \zeta. \end{aligned} \tag{3.16}$$

Now, the evaluation of the Chern polynomials on Φ^D yields representative forms of the Chern classes requested. For instance, if ∇ is a flat connection, and (e, e_*) are parallel bases, (3.16) yields

$$\Phi^D = -\frac{1}{4}(Y^{-1}\,\mathrm{d}\overline{Z}) \wedge (Y^{-1}\,\mathrm{d}Z), \tag{3.17}$$

and we see that $c_1(E)$ is represented by $(1/4\pi) \times$ (Kaehler form of $g_{\mathscr{V}}$ of (1.24)).

c. Finally, we shall discuss mappings $\varphi : N \to M$, where N is a Kaehler manifold with complex structure j and metric α, and M is a symplectic manifold with symplectic form ω, which are analogous to the harmonic mappings of riemannian geometry.

Firstly, a mapping $\varphi : N \to M$ will be called $\mathscr{J}_a$*-holomorphiable* $(a = 1, 2)$ if it can be written as $\varphi = \pi \circ \psi$ for some holomorphic mapping $\psi : (N, j) \to (\mathscr{T}(M, \omega), \mathscr{J}_a)$ where $\mathscr{J}_a$ is associated by (1.10) to some symplectic torsionless connection ∇ on M. The characterization of this property is obtainable as in [10], [11]. Namely, we must have

$$\psi_* \circ j = \mathscr{J}_a \circ \psi_* \qquad (\psi_* = \mathrm{d}\psi). \tag{3.18}$$

This relation has a horizontal component, which we obtain by applying the differential π_*, and by using (1.15) for both J and j. The result is

$$J^{\pm}\varphi_*(j^+u) = 0 \qquad (J = \psi(pr_N u)), \tag{3.19}$$

where $+$ is for $a = 1$, and $-$ is for $a = 2$. The expression of the vertical component of (3.18) follows by using the relation

$$(\psi_*(u))^v = (\varphi^{-1}\nabla)_u J \qquad (u \in TN), \tag{3.20}$$

which is provided to us by (1.34), and where $\varphi^{-1}\nabla$ is the pullback of ∇ to the vector bundle $\varphi^{-1}(TM) = \psi^{-1}(E)$. The two sides of (3.20) are endomorphisms of this latter bundle. Using again (1.15), and taking u to be successively of the complex type (1, 0) and (0, 1), the vertical component of (3.18) mentioned above becomes (after a complex conjugation)

$$J^-[(\varphi^{-1}\nabla)_{j^-u}(J^+Y)] = 0, \tag{3.21}$$

where $J = \psi(pr_N u)$, and Y is a cross-section of $\varphi^{-1}(TM)$.

Furthermore, we may see φ_* as a cross-section of $T^*N \otimes \varphi^{-1}(TM)$ with the local components $\varphi_p^\lambda = \partial x^\lambda/\partial\tau^p$, where φ has the local equations $x^\lambda = x^\lambda(\tau^p)$, and φ_* has a covariant derivative $\widetilde{\nabla}\varphi_* \overset{\text{def}}{=} \nabla^\alpha \otimes (\varphi^{-1}\nabla)\varphi_*$, where ∇^α is the Levi-Civita connection of the Kaehler metric α of N. Then we may define the *pseudo-tension field* [15]

(3.22) $$t^{\lambda}(\varphi) = \alpha^{pq}\tilde{\nabla}_p \varphi^{\lambda}_q .$$

(Notice that we have [15]

(3.23) $$\tilde{\nabla}_p \varphi^{\lambda}_q = \frac{\partial \varphi^{\lambda}_q}{\partial \tau^p} - \overset{\alpha}{\Gamma}{}^{s}_{qp}\varphi^{\lambda}_s + \Gamma^{\lambda}_{\mu\nu}\varphi^{\mu}_q \varphi^{\nu}_p,$$

where $\Gamma^{\cdot}_{\cdot\cdot}$, $\overset{\alpha}{\Gamma}{}^{\cdot}_{\cdot\cdot}$ are the connection coefficients of ∇, ∇^{α} respectively, and the symmetry of $\Gamma^{\cdot}_{\cdot\cdot}$ implies $\tilde{\nabla}_p \varphi^{\lambda}_q = \tilde{\nabla}_q \varphi^{\lambda}_p$). The mapping φ is said to be *pseudoharmonic* if $t(\varphi) = 0$. The interest of this notion (if any) relies on the fact that $t(\varphi) = 0$ is an elliptic system of equations. Hopefully, a good understanding of such pseudoharmonic mappings could yield progress in symplectic geometry, in the way Gromov's pseudoholomorphic curves [6] did. Particularly, if M is a 2-dimensional manifold with ω defined by (2.39), (2.40), and if ∇ is the Levi-Civita connection of the metric (2.39), then pseudoharmonic means in fact harmonic.

Now, the relations between the twistor spaces and pseudoharmonic mappings is exactly the same as in riemannian geometry, namely [10], [11]:

3.2. PROPOSITION. *Let $\varphi : N \to M$ be a mapping from the Kaehler manifold (N, j, α) to the symplectic manifold (M, ω) endowed with the torsionless symplectic connection ∇. Then, if either* i) *φ is $\mathscr{J}_2$-holomorphiable or,* ii) *φ is horizontally $\mathscr{J}_1$-holomorphiable (i.e., $\varphi = \pi \circ \psi$ where $\psi : N \to \mathscr{T}$ is $\mathscr{J}_1$-holomorphic and with* $(\operatorname{im} \psi_*)^v = 0$), *$\varphi$ is a pseudoharmonic mapping.*

The proof is like for Theorems 5.6 and 5.7 of [10]: one uses complex analytic coordinates σ^u, $\bar{\sigma}^v$ on N (instead of τ^p), which, by (3.23) yields

(3.24) $$\tilde{\nabla}_{\bar{u}} \varphi^{\lambda}_v = \left[(\varphi^{-1}\nabla)_{\frac{\partial}{\partial\bar{\sigma}^u}}\left(\varphi_*\left(\frac{\partial}{\partial\sigma^v}\right)\right)\right]^{\lambda} - \left[\varphi_*\left(\nabla^{\alpha}_{\frac{\partial}{\partial\bar{\sigma}^u}}\frac{\partial}{\partial\sigma^v}\right)\right]^{\lambda} = \left[(\varphi^{-1}\nabla)_{\frac{\partial}{\partial\bar{\sigma}^u}}\varphi_*\left(\frac{\partial}{\partial\sigma^v}\right)\right]^{\lambda}$$

$\left(\text{we used that } \nabla^{\alpha}_{\frac{\partial}{\partial\bar{\sigma}^u}}\frac{\partial}{\partial\sigma^v} = 0 \text{ in Kaehler geometry}\right)$; then

(3.25) $$t^{\lambda}(\varphi) = 2\alpha^{\bar{u}v}\tilde{\nabla}_{\bar{u}}\varphi^{\lambda}_v = \bar{t}^{\lambda}(\varphi),$$

and the manipulation of (3.19) and (3.21) leads from the hypotheses to the conclusion. ∎

Remark. In fact, Proposition 3.2 holds for any evendimensional manifold M

endowed with a torsionless connection ∇.

REFERENCES

[1] L. BERARD-BERGERY, T. OCHIAI: *On some generalizations of the construction of twistor spaces*, In «Global Riemannian Geometry» (T.J. Willmore and N.J. Hitchin, eds.), Ellis Horwood Ltd. Chichester, 1984, p. 52 - 59.

[2] R. DEHEUVELS: *Formes quadratiques et groupes classiques*, Presses Univ. France, Paris, 1981.

[3] M. DUBOIS-VIOLETTE: *Structures complexes au-dessus des variétés; applications*, In «Mathématique et physique. Sém. Ecole Norm. Sup. 1979 - 1982» (de Movel, Douady, Verdier, eds.), Progress in Math. 37, Birkhäuser Boston, 1983, p. 1 - 42.

[4] M. FERNANDEZ, M.J. GOTAY, A. GRAY: *Compact parallelizable four dimensional symplectic and complex manifolds and a conjecture of Thurston*, Preprint 1986.

[5] W. GREUB, S. HALPERIN, R. VANSTONE: *Connections, Curvature and Cohomology*, Vol. I, Academic Press, New York, 1972.

[6] M. GROMOV: *Pseudo holomorphic curves in symplectic manifolds*, Inventiones Math., 82 (1985), 307 - 347.

[7] S. KOBAYASHI, K. NOMIZU: *Foundations of Differential Geometry, I, II*, Intersc. Publ., New York, 1963, 1969.

[8] P. LIBERMANN: *Sur le problème d'équivalence de certaines structures infinitésimales*, Annali Mat. Pura Appl. 36 (1954), 27 - 120.

[9] N.R. O'BRIAN, J.H. RAWNSLEY: *Twistor spaces*, Annals of Global Analysis and Geometry 3 (1985), 29 - 58.

[10] J.H. RAWNSLEY: *f-structures, f-twistor spaces and harmonic maps*, In: «Geometry Seminar Luigi Bianchi II - 1984» (E. Vesentini, ed.), Lect. Notes in Math. 1164, Springer Verlag, Berlin 1985, p. 86 - 159.

[11] S. SALAMON: *Harmonic and holomorphic maps*, In: «Geometry Seminar Luigi Bianchi II - 1984» (E. Vesentini, ed.), Lect. Notes in Math. 1164, Springer Verlag, Berlin 1985, p. 161 - 224.

[12] C.L. SIEGEL: *Symplectic geometry*, Academic Press, New York, 1964.

[13] I. VAISMAN: *Cohomology and Differential Forms*, M. Dekker, Inc. New York, 1973.

[14] I. VAISMAN: *On locally conformal almost Kaehler manifolds*, Israel J. Math. 24 (1976), 338 - 351.

[15] I. VAISMAN: *A. Schwarz lemma for complex surfaces*, In Global Analysis-Analysis on Manifolds Dedicated to «Marston Morse» (Th. M. Rassias, ed.), Teubner Verlag, Leipzig, 1983, p. 305 - 323.

[16] I. VAISMAN: *Lagrangian foliations and characteristic classes*, In: «Differential Geometry» (L.A. Cordero, ed.) Research Notes in Math. 131, Pitman, Inc., London, 1985, p. 245 - 256.

[17] I. VAISMAN: *Locally conformal symplectic manifolds, Internat*, J. of Math. and Math. Sci. 8 (1985), 521 - 536.

[18] I. VAISMAN: *Symplectic Curvature Tensors*, Monatshefte für Math. 100 (1985), 299 - 327.

[19] A. WEIL: *Introduction à l'étude des variétés Kählériennes*, Hermann, Paris, 1958.

Manuscript received: October 27, 1986.